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Weak-force detection with superposed coherent states

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We investigate the utility of nonclassical states of simple harmonic oscillators, particularly a superposition of coherent states, for sensitive force detection. We find that like squeezed states, a superposition of coherent states allows displacement measurements at the Heisenberg limit. Entangling many superpositions of coherent states offers a significant advantage over a single-mode superposition state with the same mean photon number.

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I. INTRODUCTION

Nonclassical states of light have received considerable attention in the field of quantum and atom optics. Many nonclassical states of light have been experimentally produced and characterized. These states include photon number states, squeezed states, and certain entangled states. There are a number of suggested, and actual, applications of these states in quantum-information processing including: quantum cryptography [1,2], quantum teleportation [3–8], dense coding [9], and quantum communication [10–12] to name but a few. They have also been proposed for high-precision measurements such as improving the sensitivity of Ramsey fringe interferometry [13] and the detection of weak tidal forces due to gravitational radiation. In this paper, we consider how nonclassical states of simple harmonic oscillators may be used to improve the detection sensitivity of weak classical forces.

When a classical force $F(t)$ acts for a fixed time on a simple harmonic oscillator, with resonance frequency ω and mass m, it displaces the complex amplitude of the oscillator in phase space with the amplitude and phase of the displacement determined by the time dependence of the force [14]. In an interaction picture rotating at the oscillator frequency, the action of the force is simply represented by the unitary displacement operator

$$D(\alpha)=\exp(\alpha a^{\dagger}-\alpha^{*}a), \tag{1}$$

where $a,a^{\dagger}$ are the annihilation and creation operators for the single mode of the electromagnetic field satisfying $[a,a^{\dagger}]=1$, and α is a complex amplitude which determines the average field amplitude, $\langle a\rangle=\alpha$. For simplicity, we will assume that the force displaces the oscillator in a phase-space direction that is orthogonal to the coherent amplitude of the initial state, which we take to be real with no loss of generality. The displacement is thus in the momentum quadrature, $\hat{Y}=-i(a-a^{\dagger})$. To detect the force, we would need to measure this quadrature. If the oscillator begins in a coherent state $|\alpha_0\rangle$, (α_0 is real) the displacement $D(i\epsilon)$ causes the coherent state to evolve to $e^{i\epsilon\alpha_0}|\alpha_0+i\epsilon\rangle$. The signal is then measured to be $S=\langle\hat{Y}_{\rm out}\rangle=2\epsilon$, while the variance in the signal is given by $V=\langle\hat{Y}_{\rm out}^2\rangle-\langle\hat{Y}_{\rm out}\rangle^2=1$. The signal-to-noise ratio is hence

$$R=\frac{S}{\sqrt{V}}=2\epsilon, \tag{2}$$

which must be greater than unity to be resolved (the measured signal must be greater than the uncertainty of this quadrature in a coherent state). Thus, we find a standard quantum limit for the weak force detection as

$$\epsilon_L\geqslant\frac{1}{2}. \tag{3}$$

II. WEAK FORCE DETECTION WITH SQUEEZED STATES

It is well known [15] that this limit may be overcome if the oscillator is first prepared in a squeezed state (a uniquely quantum-mechanical state) for which the uncertainty in the momentum quadrature is reduced below the coherent-state level. For the case of an appropriately squeezed vacuum state

$$|\psi\rangle=\sqrt{1-|\lambda|^2}\sum_{n=0}^{\infty}\frac{\lambda^n\sqrt{(2n)!}}{n!}|2n\rangle, \tag{4}$$

where the mean photon number is given by

$$\bar{n}=\lambda^2/(1-\lambda^2) \tag{5}$$

and $\lambda=\tanh r$ (with r being the squeezing parameter). A weak force causes a displacement $D(i\epsilon/2)$ on the squeezed vacuum. In this case, the signal and variance for the measured momentum quadrature is given by [16]

$$S=\langle\hat{Y}_{\rm out}\rangle=2\epsilon, \tag{6}$$

$$V=\langle\hat{Y}_{\rm out}^2\rangle-\langle\hat{Y}_{\rm out}\rangle^2=e^{-2r}, \tag{7}$$

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and hence a signal-to-noise ratio of $R=2\epsilon e^{r}$. The minimum detectable force is given by [16]

$$\epsilon \geqslant \frac{1}{2e^{r}}, \tag{8}$$

which for large squeezing corresponds to $\epsilon_{\min}\geqslant 1/(4\sqrt{n})$. We see that squeezing provides an increased sensitivity that scales as $1/\sqrt{n}$.

Following early work by Bollinger *et al.* [17], Huelga *et al.* [13] have shown that quantum entangled states can be used to improve the sensitivity of frequency estimation using Ramsey fringe interferometry. Can entanglement be used to improve the sensitivity for force detection? To begin, let us consider an entangled state of two harmonic oscillators, the two-mode squeezed state,

$$|\psi\rangle=\sqrt{1-\lambda^{2}}\sum_{n=0}^{\infty}\lambda^{n}|n,n\rangle, \tag{9}$$

where $|n,n\rangle=|n\rangle_1\otimes|n\rangle_2$. The entanglement in this state can be seen in a variety of ways. Most obviously, it is an eigenstate of the number difference operator $a_1^{\dagger}a_1-a_2^{\dagger}a_2$, between the two modes, and in the limit of large squeezing, $\lambda\rightarrow 1$, a near eigenstate of phase sum [18]. Alternatively we can consider the correlations between quadrature phase operators. In the limit of large squeezing $(\lambda\rightarrow 1)$, the state approaches a simultaneous eigenstate of both $\hat{X}_1-\hat{X}_2$ and $\hat{Y}_1+\hat{Y}_2$, which is the kind of state considered by Einstein, Podolsky, and Rosen [19]. This kind of correlation has been exploited by Furasawa *et al.* [20] to realize an experimental teleportation protocol. With two oscillators, we need to specify how the weak force acts. We will specify that the force acts independently on each oscillator. To detect the force, consider a measurement of the joint physical quantity described by the operator $\hat{Y}_1+\hat{Y}_2$. It is then straightforward to show that the signal and variance of the measured result, after the displacement, are given by

$$S=\langle\hat{Y}_1+\hat{Y}_2\rangle=4\epsilon, \tag{10}$$

$$V=\langle(\hat{Y}_1+\hat{Y}_2)^2\rangle-\langle\hat{Y}_1+\hat{Y}_2\rangle^2=2e^{-2r}, \tag{11}$$

which gives a signal-to-noise ratio of $R=2\sqrt{2}\epsilon e^{r}$. The minimum detectable force is then $\epsilon\geqslant 1/(2\sqrt{2}e^{r})$ which is a $\sqrt{2}$ improvement over the single-mode squeezed state. For large squeezing, the minimum detectable force can be expressed in terms of the total mean photon number for both modes. In this limit, $\epsilon_{\min}\approx 1/(4\sqrt{n_{\mathrm{tot}}})$. This is the same scaling as we found for a single-mode squeezed state. The apparent improvement due to entanglement is simply a reflection of the fact that we have a two-mode resource with double the mean photon number.

For the two-mode squeezed state with the measurement scheme chosen, there is a simple way to understand this result. The entangled two-mode squeezed state (9) is easily disentangled by the application of a unitary operator of the form $U=\exp[-i\pi(a_1^{\dagger}a_2+a_1a_2^{\dagger})/4]$, which does not change the total energy. We will refer to this unitary transformation as the beam splitter transformation, as in the case that the two oscillator modes correspond to optical field modes, this transformation describes the scattering matrix of an optical beam splitter. The resulting state becomes a (disentangled) product state of two single-mode squeezed states [as in Eq. (4)]. The weak force now acts to displace each of the single-mode squeezed states, each of which may be used to achieve the squeezed state limit for displacement detection. As there are two realizations of the measurement scheme, there will be an additional $1/\sqrt{2}$ improvement in sensitivity simply from classical statistics. It is thus inaccurate to attribute the improved force sensitivity of a two-mode squeezed state to entanglement when $\hat{Y}_1+\hat{Y}_2$ measurements are performed. In assessing the limits to force detection using entangled states of N harmonic oscillators we thus need to consider if any apparent improvement could have been achieved simply by using N copies of an appropriate nonclassical state of a single-harmonic oscillator.

Of course it may not always be so obvious to transform an entangled state to a product of nonclassical states. Consider an entangled state of the form

$$|\Psi\rangle=\sum_{n=0}^{\infty}c_n|n,n\rangle. \tag{12}$$

This state is correlated in number, but unlike the two-mode squeezed state, it is not necessarily a near eigenstate of phase sum. If we consider a measurement of Y_1+Y_2 as we did previously, the signal and variance after the displacement are

$$S=4\epsilon, \tag{13}$$

$$V=2(1+\langle a^{\dagger}a+b^{\dagger}b\rangle-\langle a^{\dagger}b^{\dagger}+ab\rangle), \tag{14}$$

which gives an improvement in the signal-to-noise ratio when $\langle a^{\dagger}a+b^{\dagger}b\rangle<\langle a^{\dagger}b^{\dagger}+ab\rangle$. A state like this, with a correlated photon number, is the pair-coherent (or "circle") state given by [21,22]

$$|\mathrm{circle}\rangle_m=\mathcal{N}\int_0^{2\pi}|\alpha e^{i\zeta}\rangle_a|\alpha e^{-i\zeta}\rangle_b d\zeta, \tag{15}$$

where $|\ldots\rangle_a$ and $|\ldots\rangle_b$ represent coherent states in the modes $\hat{a}$ and $\hat{b}$. $\mathcal{N}$ is a normalization coefficient and α the amplitude of the coherent state. This state can be written in the form (12) with

$$c_n=\frac{1}{\sqrt{I_0(2\alpha)}}\frac{\alpha^{n}}{n!}. \tag{16}$$

Here, I_0 is a zeroth-order modified Bessel function. This state cannot be separated into product states via beam splitter transformations. It is easily shown that the minimum detectable force occurs when

$$\epsilon_{\min}=\frac{1}{2}\sqrt{\frac{1}{2}+\bar{n}-\alpha}, \tag{17}$$

with the mean photon number being given by $\bar{n}=\alpha I_1(2\alpha)/I_0(2\alpha)$. A small improvement is seen for all α, with the minimum occurring at $\alpha=0.85$ ($\epsilon_{\min}=0.221\,108$). As $\alpha\to\infty$, we have $\epsilon_{\min}\to 0.25$. In this optimal region the mean photon number is small. The measurement of Y_1+Y_2 is not optimal however because it is not a near eigenstate.

It is likely that one can achieve a significantly better sensitivity if one changes the measurement quantity from Y_1+Y_2 to a quantity that is a near eigenstate of Eq. (12). For these correlated photon number systems, this could require a measurement of the photon number difference of Eq. (12) which with current technology is quite unpractical.

III. WEAK FORCE DETECTION WITH CAT STATES

Let us now turn our attention to a less straightforward example. In the previous example, two entangled harmonic modes, the two-mode squeezed state, gave an improvement in the signal-to-noise ratio (compared to a single mode) of $1/\sqrt{2}$. With an entangled state comprised of more modes, an even better improvement may be achievable. However, there is no simple way to generalize the two-mode squeezed state to give an entangled state of many modes. We now consider another class of nonclassical states, based on a coherent superposition of coherent states (cat states), which can be entangled over N modes.

Consider N harmonic oscillators prepared in the cat state

$$|\psi\rangle_N=\mathcal{N}_+(|\alpha,\alpha,\ldots,\alpha\rangle+|-\alpha,-\alpha,\ldots,-\alpha\rangle), \tag{18}$$

where

$$|\alpha,\alpha,\ldots,\alpha\rangle=\Pi_k^{\otimes N}|\alpha\rangle_k \tag{19}$$

is the tensor product of coherent states and $\mathcal{N}$ is the normalization constant given by

$$\mathcal{N}=\frac{1}{\sqrt{2+2e^{-2N|\alpha|^2}}}. \tag{20}$$

We take α to be real for convenience. For $\alpha\gg 1$, this normalization constant approaches $1/\sqrt{2}$, and we henceforward make this assumption. Parkins and Larsabal [23] recently suggested how this highly entangled state might be formed in the context of cavity QED and quantized motion of a trapped atom or ion.

To begin our consideration of these states, let us consider the case of a single oscillator ($N=1$)

$$|\phi\rangle=\frac{1}{\sqrt{2}}(|\alpha\rangle+|-\alpha\rangle), \tag{21}$$

where the mean photon number is given by $\bar{n}=|\alpha|^2$. When a weak classical force acts on the state in Eq. (21), it is displaced by

$$\begin{aligned}|\phi\rangle_{\text{out}}&=\frac{1}{\sqrt{2}}(e^{-i\,\text{Im}(\alpha\beta^*)}|\alpha+\beta\rangle+e^{i\,\text{Im}(\alpha\beta^*)}|-\alpha+\beta\rangle)\\&\approx\frac{1}{\sqrt{2}}(e^{i\theta}|\alpha\rangle+e^{-i\theta}|-\alpha\rangle)=\cos\theta|+\rangle+i\sin\theta|-\rangle,\end{aligned} \tag{22}$$

where $\theta=-2\,\text{Im}(\alpha\beta^*)$ and we have defined the even- ($|+\rangle$) and odd-parity ($|-\rangle$) eigenstates

$$|\pm\rangle=\frac{1}{\sqrt{2}}(|\alpha\rangle\pm|-\alpha\rangle). \tag{23}$$

Our problem is thus reduced to finding the optimal readout for the rotation parameter θ for a two-dimensional submanifold of parity eigenstates. The rotation is described by the unitary transformation

$$U(\theta)=\exp(i\theta\hat{\sigma}_x), \tag{24}$$

where $\hat{\sigma}_x=|+\rangle\langle-|+|-\rangle\langle+|$ is a Pauli matrix.

The objective is now to find an optimal measurement scheme to estimate the rotation parameter θ and thus the force parameter ϵ. The maximum sensitivity will occur when $\theta=-\text{Im}(\alpha\beta^*)$ is maximized for a given displacement. Having chosen α real, θ is maximized by choosing β purely imaginary. This corresponds to a displacement $D(\beta)$ entirely in the momentum quadrature. Setting $\beta=i\epsilon$, we have $\theta=\epsilon\alpha$. The theory of optimal parameter estimation [24] indicates that the limit on the precision with which the rotation parameter can be determined is

$$(\delta\theta)^2\geq\frac{1}{\text{Var}(\hat{\sigma}_x)_{\text{in}}}, \tag{25}$$

where $\text{Var}(\hat{\sigma}_x)_{\text{in}}$ is the variance in the generator of the rotation in the input state $|+\rangle$, which is simply unity. Thus, we find that uncertainty on the force parameter is bounded below by $\delta\epsilon\geq 1/(2\alpha)$. It thus follows that the minimum detectable force is $\epsilon_{\min}\geq 1/(2\alpha)$, which may be written in terms of the total mean excitation number of the input state as

$$\epsilon\geq\frac{1}{2\sqrt{\bar{n}}}, \tag{26}$$

where the mean photon number $\bar{n}=|\alpha|^2$. This measurement is at the Heisenberg limit. Comparison with the result for the single-mode squeezed state shows a similar dependence on the mean excitation number, however, the squeezed state sensitivity is better by a factor 1/2.

We can now consider a two-mode entangled cat state.

$$|\psi\rangle_1=\mathcal{N}(|\alpha,\alpha\rangle+|-\alpha,-\alpha\rangle). \tag{27}$$

However, this state is easily disentangled with the unitary transformation

$$U(\pi/2)=\exp\left[-i\frac{\pi}{2}(a_1^\dagger a_2+a_1 a_2^\dagger)\right] \tag{28}$$

(for a quantum optical realization, this is a 50:50 beam splitter) to produce the separable state

$$|\tilde{\psi}\rangle_1=\mathcal{N}_1\mathcal{N}_2(|\alpha\rangle_1+|-\alpha\rangle_1)\otimes(|\alpha\rangle_2+|-\alpha\rangle_2). \tag{29}$$

As in the case for squeezed states, we only need consider the force detection sensitivity for the state of a single oscillator. The minimum detectable force is given by

$$\epsilon\geqslant\frac{1}{2\sqrt{2\bar{n}}}. \tag{30}$$

Here, we see the $\sqrt{2}$ improvement from classical averaging. For the N-mode state given by Eq. (18), a linear transformation also exists to transform the N-mode entangled state to a product state of single-mode cat states. In this case, the minimum detectable force using N modes, each prepared in cat state with amplitude α, is

$$\epsilon_{\min}>\frac{1}{2\sqrt{N\bar{n}}}. \tag{31}$$

As each mode has a mean photon number given by $\bar{n}=\alpha^2$, the total mean photon number used in the entire experiment is $\bar{n}_{\rm tot}=N\alpha^2$, the minimum detectable force can be written as $\epsilon_{\min}>1/\sqrt{\bar{n}_{\rm tot}}$. We see from here that there is no real advantage in using entangled states with the measurement protocol outlined, as the improvement is only the standard statistical improvement that one gets from multiple copies of a single-mode cat state produced by disentangling the state.

IV. ENTANGLED CAT STATES

A question that should be asked is whether both entanglement and collective measurements allow one to increase the sensitivity of this displacement measurement past the limits shown above? To address this question, let us consider again the N-mode entangled cat state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|\alpha,\alpha,\ldots,\alpha\rangle+|-\alpha,-\alpha,\ldots,-\alpha\rangle), \tag{32}$$

where the total photon number of the entire state is $n_{\rm tot}=N\alpha^2$. The weak force acts simultaneously on all modes of this N-party entangled cat state. It causes a displacement $D(i\epsilon)$ on each mode in Eq. (32) resulting in the state

$$|\psi(\theta)\rangle=\frac{e^{iN\theta}}{\sqrt{2}}|\alpha+i\epsilon,\alpha+i\epsilon,\ldots,\alpha+i\epsilon\rangle + \frac{e^{-iN\theta}}{\sqrt{2}}|-\alpha+i\epsilon,-\alpha+i\epsilon,\ldots,-\alpha+i\epsilon\rangle, \tag{33}$$

where $\theta=\epsilon\alpha$. The theory of optimal parameter estimation indicates that the limit on the precision with which the rotation parameter is given by Eq. (25) but where $\sigma_x=\Sigma_{i=1}^N\sigma_{x_i}$. The uncertainty in this force parameter is hence bounded by

$$\epsilon=\frac{1}{2N\alpha}=\frac{1}{\sqrt{4Nn_{\rm tot}}} \tag{34}$$

and is at the Heisenberg limit. We observe a critically important extra $\sqrt{N}$ improvement due to the entangled state and collective measurement (projective measurements onto $|\alpha,\alpha,\ldots,\alpha\rangle-|-\alpha,-\alpha,\ldots,-\alpha\rangle$) which can be seen over N individual copies of the state $|\alpha\rangle+|-\alpha\rangle$, or a single-mode state $|\sqrt{n_{\rm tot}}\rangle+|-\sqrt{n_{\rm tot}}\rangle$. For a large and finite $n_{\rm tot}$ it seems optimal that one should create a highly entangled cat state with as many modes as possible while maintaining $\alpha\gg1$.

In our consideration so far we have not considered the effects of loss or decoherence on these highly nonclassical states. Whether we are considering highly entangled cat states or large-amplitude single-mode cat states these are all extremely sensitive to small amounts of loss and decoherence. Error correction and avoidance techniques can be employed to reduce these effects but are beyond the scope of this paper.

V. GENERALIZED CAT STATES

In the example just discussed, maximum sensitivity required the classical force to displace the cat states in a direction orthogonal to the phase of the superposed coherent amplitudes. In general, there is no way to arrange this beforehand, as the phase of the displacement depends on an unknown time dependence of the classical force. However, a simple generalization of the previous cat states can be used to relax this constraint. Note that the cat states are parity eigenstates and are thus the conditional states resulting from a measurement of the number operator modulo 2, $\hat{n}_2=a^\dagger a \bmod 2$, on an input state $|\alpha\rangle$ with α real. We are thus led to consider the conditional states for measurements of $\hat{n}_K=a^\dagger a \bmod K$. Such states have previously been considered by Schneider *et al.* [25]. Given a result $\nu=0,1,\ldots,K-1$ for such a measurement, the conditional (unnormalized) states are

$$|K,\nu\rangle=\sum_{\mu=0}^{K-1}\exp\left[\frac{2\pi i\mu\nu}{K}\right]|\alpha e^{2\pi i\mu/K}\rangle, \tag{35}$$

which are eigenstates of $e^{i2\pi a^\dagger a/K}$ with eigenvalues $e^{-i2\pi\nu/K}$.

The case of $K=4$ has recently been considered by Zurek [26] in the context of decoherence and quantum chaos. Assume that the oscillator is initially prepared in the state

$$|\psi\rangle_{in}=|4,0\rangle=|\alpha\rangle+|i\alpha\rangle+|-i\alpha\rangle+|-\alpha\rangle \tag{36}$$

with α real. Under the action of a weak force characterized by a complex amplitude displacement β, the output state is

$$|\psi\rangle_{\text{out}}=e^{i\theta}|\alpha\rangle+e^{i\phi}|i\alpha\rangle+e^{-i\phi}|-i\alpha\rangle+e^{-i\theta}|-\alpha\rangle, \tag{37}$$

where $\theta=2\alpha\,\mathrm{Im}(\beta)$ and $\phi=2\alpha\,\mathrm{Re}(\beta)$. The state now carries information on both the real and imaginary components of the displacement due to the force which may be extracted by measuring the projection operator onto the initial state. In the limit that $K\gg|\alpha|^2\gg 1$, the initial conditional state is simply the vacuum state and we recover the usual standard quantum limit for force detection by number measurement [15].

VI. DISCUSSION AND CONCLUSION

We now compare our results to the study of Ramsey fringe interferometry introduced by Bollinger *et al.* [17] and discussed by Huelga *et al.* [13]. In Ramsey fringe interferometry, the objective is to detect the relative phase difference between two superposed states $\{|0\rangle,|1\rangle\}$ that form a basis for a two-dimensional Hilbert space. These states could be the ground and excited states of an electronic dipole transition. The problem reduces to a quantum parameter estimation problem. The unitary transformation which induces a relative phase in the specified basis is $U(\theta)=\exp[i\theta\hat{Z}]$ where $\hat{Z}=|1\rangle\langle 1|-|0\rangle\langle 0|$. We are free to choose the input state $|\psi\rangle_i$ and the measurement we make on the output state, which is described by an appropriate positive operator valued measure (POVM).

The theory of quantum parameter estimation [24] indicates in this case that we should choose the input state as $|\psi\rangle_i=(|0\rangle+|1\rangle)/\sqrt{2}$ and the optimal measurement is a projective measurement in the basis $|\pm\rangle=|0\rangle\pm|1\rangle$. The probability to obtain the result $+$ is $P(+|\theta)=\cos^2\theta$. In N repetitions of the measurement the uncertainty in the inferred parameter is

$$\delta\theta=\frac{1}{\sqrt{N}}, \tag{38}$$

which achieves the lower bound for quantum phase parameter estimation. Repeating the measurement N times is equivalent to a single-product POVM on the initial product state $\Pi_{i=1}^{N}\otimes(|0\rangle_i+|1\rangle_i)/\sqrt{2}$. However, it was first noted by Bollinger *et al.* [17] that a more effective way to use the N level systems is to first prepare them in the maximally entangled state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2\ldots|0\rangle_N+|1\rangle_1|1\rangle_2\ldots|1\rangle_N) \tag{39}$$

and subjecting the entire state to the unitary transformation $U(\theta)=\Pi_{i=1}^{N}\exp(-i\theta\hat{Z}_i)$, the uncertainty in the parameter estimation then achieves the Heisenberg lower bound of

$$\delta\theta=\frac{1}{N}. \tag{40}$$

Briefly, let us instead consider $N/2$ maximally entangled pairs. In this case, we can combine Eq. (40) at $N=2$ with the square-root statistical benefit of $N/2$ repetitions. This yields $\delta\theta=\frac{1}{2}\sqrt{2/N}=1/\sqrt{2N}$ indicating that pairwise entanglement yields only a margined benefit compared to full N-wise entanglement for the phase estimation.

We will now show that the entangled state in Eq. (39) is in fact a cat state for a collective operator algebra. The Hilbert space of N two-level systems is the tensor product space of dimension 2^N. The entangled state in Eq. (39) however resides in a lower-dimensional subspace of permutation symmetric states [27]. These states constitute an $N+1$-dimensional irreducible representation of SU(2) with infinitesimal generators defined by

$$\hat{J}_z=\frac{1}{2}\sum_{i=1}^{N}\hat{Z}_i,\quad \hat{J}_y=\frac{1}{2}\sum_{i=1}^{N}\hat{Y}_i,\quad \hat{J}_x=\frac{1}{2}\sum_{i=1}^{N}\hat{X}_i, \tag{41}$$

where $\hat{Z}_i=|1\rangle_i\langle 1|-|0\rangle_i\langle 0|$, $\hat{X}_i=|1\rangle_i\langle 0|+|0\rangle_i\langle 1|$, $\hat{Y}_i=i|1\rangle_i\langle 0|-i|0\rangle_i\langle 1|$. The Casimir invariant is $\hat{J}^2=\hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2$ with eigenvalue $N/2\,(N/2+1)$. The operator $\hat{J}_z$ has eigenvalues $m=-N/2,-N/2+1,\ldots,N/2$, which is one half the difference between the number of zeros and ones in an eigenstate. It is more convenient to use the eigenstates $|m\rangle_{N/2}$ of these commuting operators as basis states in the permutation symmetric subspace. In this notation, the entangled state defined in Eq. (39) may be written

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|-N/2\rangle_{N/2}+|N/2\rangle_{N/2}). \tag{42}$$

In this form, we can regard the state as an SU(2) "cat state" for N two-level atoms. Hence, it is straightforward to see that a single 2^N-level atom can achieve the same frequency sensitivity. Their equivalence can be also be understood by noting that the sensitivity of such frequency measurements is proportional to the energy difference of the states involved. What entanglement allows is for one to create an effective state without the need of resorting to create a superposition between a certain ground state and a highly excited one.

A closer atomic analogy to a single-mode cat state would be a cat state for a single N-level electronic system. For example, we could consider the unnormalized state defined on a hyperfine manifold with quantum number F, $|F\rangle_F+|-F\rangle_F$. Such states have been considered in Ref. [28]. A similar state could also be generated for the large magnetic molecular systems considered in Refs. [29–31]. The key is in how the resources can be distributed and what types of measurements one is trying to achieve. If the single molecule can only be prepared with a certain N, then an advantage can be gained for frequency measurements by entangling the state of many single molecule systems [31,32]. However, if we restrict the total system to having a fixed N and we have enough control so as to be able to prepare the system either as a single large SU(2) molecular state or many entangled smaller molecular states, then the same sensitivity is achieved for the high-precision frequency measurement (this was not the case for weak force measurements).

To conclude, we have in this paper shown how superpositions of coherent states can be used to achieve extremely sensitive force detection. For a single-mode state $|\alpha\rangle+|-\alpha\rangle$ we have found that the minimum detectable displacement for weak force measurements scales inversely proportional to the square root of the mean photon number of the superposition of coherent states. This is the same scaling obtained by a single-mode squeezed state and achieves the Heisenberg limit for single-mode displacement measurements.

What is potentially more interesting is that if we take a number of individual copies of a single-mode cat state then we still achieve the inverse square-root scaling with total mean photon number (hence, effectively allow one to increase the mean number of particles). If one starts with an N-mode entangled cat state, then simple linear transformation can be used to turn this state into N copies of a single-mode cat state and hence achieve the $\epsilon_{\min}\sim 1/\sqrt{n_{\rm tot}}$ sensitivity. This however is not optimal as it does not achieve the Heisenberg limit for multiple modes. To achieve this limit for weak force detection, one must use both an entangled N-mode cat state and a joint collective measurement (between the various modes). Entanglement is the critical resource to achieve the best sensitivity for a fixed $n_{\rm tot}$. On the other hand, we have shown for frequency (or phase) measurements that the sensitivity previously offered by entangling N two-level atoms can be achieved with a single 2^N-level atom. The key is that the sensitivity for the frequency measurement is proportional to the energy difference of the states involved and both the entangled resource and the superposition resource have the same energy difference. Entanglement allows one to create an effective state without the need of resorting to create a superposition between the certain ground state and a highly excited one.

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